CONTINUOUS SELECTIONS AND FIXED POINTS OF MULTI-VALUED MAPPINGS ON NON-COMPACT OR NON-METRIZABLE SPACES

LAI-JIU LIN, NGAI-CHING WONG AND ZENN-TSUEN YU

ABSTRACT. In this paper, we obtain several new continuous selection theorems for multi-valued mappings on completely regular spaces and fixed point theorems for multi-valued maps on non-metrizable spaces. They, in particular, provide a partial solution of a conjecture of X. Wu.

1. INTRODUCTION

In [4, 5], Browder first used continuous selection theorem to prove the Fan-Browder fixed point theorem. Later, Yannelis and N. D. Prabhakar [17], Ben-El-Mechaiekh [2, 3], Ding, Kim and Tan [8], Horvath [11], Wu [16, 15], Park [12, 13], and many others, established several continuous selection theorems with applications. We note that all the continuous selection theorems studied by the above authors, the multi-valued maps are defined on a compact or paracompact space. In [17], Yu and Lin studied continuous selections of multi-valued mappings defined on noncompact spaces, but they assume some kind of coercivity conditions instead.

In this paper, we establish a continuous selection theorem for a multi-valued map defined on a completely regular topological space. We do not assume the compactness of its domain.

In the second part of this paper, we discuss collectively fixed points of lower semicontinuous multi-valued maps. Recently, many authors studied fixed point theorems of lower semicontinuous multi-valued maps, see for example [14, 6, 15, 1]. In particular, Wu established the following one.

Theorem 1.1 ([15]). *Let X be a nonempty subset of a Hausdorff locally convex topological vector space, let D be a nonempty compact metrizable subset of X, and let $T : X \to 2^D$ be a multi-valued mapping with the following properties:*

(a) $T(x)$ is a nonempty closed convex set for each x in X;

2000 *Mathematics Subject Classification.* 54C65, 46H10, 54H25.

Key words and phrases. multi-valued mappings, continuous selections, fixed points.

(b) T is lower semicontinuous.

Then there exists a point $\bar{x}$ in D such that $\bar{x} \in T(\bar{x})$.

Wu conjectured in [15] that the conclusion of Theorem 1.1 remains true even if the metrizability condition of D is dropped. In this paper, we shall use the approximate continuous selection theorem of Deutsch and Kenderov [7] (see also [19]) to establish an approximate fixed point theorem for a sub-lower semicontinuous multi-valued map. This gives rise to a partial solution of the conjecture of Wu [15]. We shall also provide a simple proof of a Himmelberg type collectively fixed point theorem. We remark that our results differ from the approximate fixed point theorem recently established by Park [13].

We would like to thank the referee for many helpful suggestions on improving the presentation and the bibliography in this paper.

2. Preliminaries

Let X and Y be topological spaces. A multi-valued map $T : X \to 2^Y$ is a map from X into the power set 2^Y of Y. Let $T^{-1} : Y \to 2^X$ be defined by the condition that $x \in T^{-1}y$ if and only if $y \in T(x)$. Recall that

(a) T is said to be *closed* if its graph $G_r(T) = \{(x, y) : x \in X, y \in T(x)\}$ is closed in the product space $X \times Y$;

(b) T is said to be *upper semicontinuous* (in short, u.s.c.) at x if for every open set V in Y with $T(x) \subset V$, there exists a neighborhood $W(x)$ of x such that $T(W(x)) \subset V$; T is said to be u.s.c. on X if T is u.s.c. at every point of X;

(c) T is said to be *lower semicontinuous* (in short, l.s.c.) at x if for every open neighborhood $V(y)$ of every y in $T(x)$, there exists a neighborhood $W(x)$ of x such that $T(u) \cap V(y) \neq \emptyset$ for all u in $W(x)$; T is said to be l.s.c. on X if T is l.s.c. at every point of X;

(d) In case Y is a topological linear space, T is said to be *sub-lower semicontinuous* (see, e.g., [19]) at an x in X if for each neighborhood V of 0 in Y, there is a z in $T(x)$ and a neighborhood $U(x)$ of x in X such that $z \in T(y)+V$ for each y in $U(x)$; T is said to be sub-lower semicontinuous on X if T is sub-lower semicontinuous at every point of X. It is plain that if T is lower semicontinuous at x, then T is sub-lower semicontinuous at x;

The following lemmas are needed in this paper.

Lemma 2.1 (Deutsch and Kenderov [7]). *Let X be a paracompact topological space, let Y be a locally convex topological linear space, and let $F : X \to 2^Y$. Then F is* *sub-lower semicontinuous if and only if for each neighborhood V of 0 in Y, there is a continuous function $f : X \to Y$ such that $f(x) \in F(x) + V$ for each x in X.*

Lemma 2.2 (Yuan [18])**.** *Let X be a topological space, let Y be a nonempty subset of a topological vector space with a base $\mathcal{B}$ for the zero neighborhoods, and let $F : X \to 2^Y$. For each V in $\mathcal{B}$, define $F_V : X \to 2^Y$ by*

$$F_V(x) = (F(x) + V) \cap Y, \quad \forall x \in X.$$

Write $\bar{y} \in \overline{F}(\bar{x})$ if $(\bar{x}, \bar{y}) \in \overline{G_r F}$. Then for any $\bar{x}$ in X and $\bar{y}$ in Y, we have

$$\bar{y} \in \overline{F}(\bar{x}) \quad \textit{whenever} \quad \bar{y} \in \bigcap\nolimits_{V \in \mathcal{B}} \overline{F_V}(\bar{x}).$$

Lemma 2.3 (Himmelberg [10])**.** *Let X be a nonempty convex subset of a locally convex topological vector space. Let $T : X \to 2^X$ be an u.s.c. multi-valued map with nonempty closed convex values such that $T(X) = \cup_{x \in X} T(x)$ is contained in a compact subset of X. Then there exists an $\bar{x}$ in X such that $\bar{x} \in T(\bar{x})$.*

Lemma 2.4 (Granas [9]; see also Ding, Kim and Tan [8])**.** *Let D be a nonempty compact subset of a topological vector space. Then the convex hull $\operatorname{co} D$ of D is σ-compact and hence is paracompact.*

3. Continuous selection theorems

Note that the set $S^{-1}(y) = \{x \in X : y \in S(x)\}$ below can have empty interior for some y in K.

Theorem 3.1. *Let X be a completely regular space and let K be a nonempty subset of a Hausdorff topological vector space E. Assume a multi-valued function $S : X \longrightarrow 2^K$ satisfies the following conditions:*

(a) For each x in X, the set $S(x)$ is convex.

(b) $X = \bigcup \{\operatorname{int} S^{-1}(y) : y \in K\}$.

Then for any compact subset F of X there is an open dense subset U of X containing F such that S has a continuous selection $f : U \to K$, that is, $f(x) \in S(x)$ for all x in U.

Proof. By assumption (b), there are finitely many points $y_1, \ldots, y_n$ in K such that

$$F \subseteq \operatorname{int} S^{-1}(y_1) \cup \cdots \cup \operatorname{int} S^{-1}(y_n).$$

For each $k = 1, \ldots, n$ and x in $F \cap \operatorname{int} S^{-1}(y_k)$, there is a continuous function g_x on X such that $0 \leq g_x \leq 1$, $g_x(x) = 1$ and g_x vanishes outside $\operatorname{int} S^{-1}(y_k)$. By the compactness of F, there are finitely many g_x such that for every point in F at least one of them assumes value not less than 1/2. Summing them in an appropriate way, we will have nonnegative continuous functions $g_1, \ldots, g_n$ on X such that g_k vanishes outside $\operatorname{int} S^{-1}(y_k)$, and $\sum_{k=1}^n g_k(x) \geq 1/2$ for all x in F. Let $V = \{x \in X : \sum_{k=1}^n g_k(x) > 1/3\}$. Set $f_j(x) = g_j(x)/\sum_{k=1}^n g_k(x)$ on V, and $f_j(x) = 3g_j(x)$ on $X \setminus V$. Define a continuous function $f_V : X \longrightarrow E$ by

$$f_V(x) = \sum_{k=1}^n f_k(x) y_k, \quad \forall x \in X.$$

For each x in V and for each k with $f_k(x) \neq 0$, we have $x \in \operatorname{int} S^{-1}(y_k)$. Hence, $y_k \in S(x)$. Consequently, $f_V(x) \in \operatorname{co}(S(x)) = S(x) \subseteq K$ for all x in V. In other words, the restriction of f_V to V gives rise to a continuous selection of S on the open set V which contains F.

Denote by

$$\mathcal{W} = \{(f_W, W) : \quad \text{where } W \text{ is an open subset of } X \text{ containing } F \text{ and } f_W : W \to K \text{ gives rise to a continuous selection of } S \text{ on } W\}.$$

Then $\mathcal{W}$ is not empty as $(f_V, V) \in \mathcal{W}$. Order $\mathcal{W}$ by extension and we get a non-empty partially ordered set. In other words, $(f_W, W) \leq (f_V, V)$ if $W \subseteq V$ and $f_{V|_W} = f_W$. Applying Zorn's Lemma, we get a maximal element (f_U, U) of $\mathcal{W}$.

The last step is to verify that U is dense in X. Suppose not and there were an x in X outside the closure of U. Let $x \in \operatorname{int} S^{-1}(y)$ for some y in K. By setting $f_{|W} \equiv y$, we get a continuous selection of S on an open neighborhood W of x disjoint from U by restriction. Then the union $f_{U \cup W} : U \cup W \longrightarrow K$ defined in a natural way provides a contradiction to the maximality of (f_U, U). □

We call a topological space X *residually paracompact* if for every open dense subset U of X the complement $X \setminus U$ is paracompact.

Theorem 3.2. *In addition to the conditions (a) and (b) in Theorem 3.1, if we assume further that*

(c) X is residually paracompact.

Then there is a continuous function $f : X \to K$ such that $f(x) \in S(x)$ for all x in X.

Proof. It follows from Theorem 3.1 that there is a continuous function $f_U : U \longrightarrow K$ defined on an open dense subset U of X with $f_U(x) \in S(x)$ for all x in U. For each z in $X \setminus U$, there is a y in K such that $z \in \operatorname{int} S^{-1}(y)$ by condition (b). By setting $f_{|W_z} \equiv y$ we get a continuous selection of S on an open neighborhood W_z of z. The paracompactness of $X \setminus U$ ensures it has a locally finite covering by open sets in X, each of which is contained in some W_z. Adding one more open set U, we have a locally finite open covering of X. This provides us with a family $\{g_\lambda\}_\lambda$ of nonzero continuous functions from X into $[0,1]$ dominated by the open sets W_z and U such that $g_\lambda(x) = 0$ for all but finitely many λ's and $\sum_\lambda g_\lambda(x) = 1$ for all x in X. If g_λ vanishes outside U, we set $f_\lambda = f_U$. Otherwise, we fix a choice of z such that g_λ vanishes outside W_z, and set $f_\lambda = f_{W_z}$. Define $f : X \longrightarrow K$ by

$$f(x) = \sum_\lambda g_\lambda(x) f_\lambda(x), \quad \forall x \in X.$$

For each x in X, only finitely many $g_\lambda(x)$'s are non-zero in the sum, and the nonzero terms give rise to a convex combination of points in the convex set $S(x)$. Thus $f(x) \in S(x)$ for all x in X. □

It is easy to see that the following corollary follows from Theorem 3.1.

Corollary 3.3. *The conclusion of Theorem 3.1 remains true if the conditions (a) and (b) are replaced by*

(a)′ for each x in X, the set $S(x)$ is a nonempty convex set;
(b)′ for each y in K, the set $S^{-1}(y)$ is open.

Remark 3.4. *Corollary 3.3 implies Theorem 3.1.*

Proof. Let $T : X \to 2^K$ be defined by

$$T(x) = \{y \in K : x \in \operatorname{int} S^{-1}(y)\}.$$

Then $T^{-1}(y) = \operatorname{int} S^{-1}(y)$ is open for each y in K. By (b), for each x in X, there exists y in K such that $x \in \operatorname{int} S^{-1}(y)$. Therefore $y \in T(x) \neq \emptyset$ for each x in X. Let $H : X \to 2^K$ be defined by $H(x) = \operatorname{co} T(x)$. Then $H(x)$ is nonempty for each x in X. and $H^{-1}(y)$ is open for each y in K. By Corollary 3.3, there is an open dense subset U of X, containing any but fixed compact set D, and there is a continuous function $f : U \to K$ such that $f(x) \in H(x) = \operatorname{co} T(x) \subset S(x)$ for all x in U. □

4. Fixed Point Theorems

Theorem 4.1. *For each i in a nonempty index set I, let X_i be a nonempty convex subset of a Hausdorff locally convex topological vector space E_i, and let D_i be a compact subset of X_i. Let $X = \prod_{i\in I} X_i$ be the product space. Let $F_i : X \to 2^{D_i}$ be sub-lower semicontinuous with nonempty convex values. Then for every neighborhood V_i of 0 in E_i, there exists a point $\bar{x}_V = (x_{V_i})$ in $D = \prod_{i\in I} D_i$ such that $(\bar{x}_{V_i} + V_i) \cap F_i(\bar{x}_V) \neq \emptyset$ for all i in I.*

Proof. Given a neighborhood V_i of zero in E_i for each i in I. Fix any i in I. There exists an absolutely convex neighborhood W_i of 0 such that $W_i \subset V_i$. Note that D is a compact subset of X. By Lemma 2.4, $\operatorname{co} D$ is a paracompact subset of X. Since $F_i : X \to 2^{D_i}$ is a sub-lower semicontinuous multi-valued map with nonempty convex values, by Lemma 2.1 there exists a continuous function $f_i : \operatorname{co} D \to D_i$ such that

$$f_i(x) \in (F_i(x) + W_i) \cap D_i \quad \text{for each} \quad x \in \operatorname{co} D.$$

Define $f : \operatorname{co} D \to D$ by $f(x) = \prod_{i\in I} f_i(x)$ for x in $\operatorname{co} D$. By Himmelberg fixed point theorem (Lemma 2.3), there exists an $\bar{x}_V = (\bar{x}_{V_i})_{i\in I}$ in $\operatorname{co} D$ such that $\bar{x}_V = f(\bar{x}_V) = \prod_{i\in I} f_i(\bar{x}_V)$. That is, $\bar{x}_{V_i} = f_i(\bar{x}_V) \in (F_i(\bar{x}_V) + W_i) \cap D_i$. Thus, $\bar{x}_{V_i} \in D_i$ and $(\bar{x}_{V_i} + W_i) \cap F_i(\bar{x}_V) \neq \emptyset$ for all i in I. Since $W_i \subset V_i$, we have $(\bar{x}_i + V_i) \cap F_i(\bar{x}_V) \neq \emptyset$ for all i in I. $\square$

Theorem 4.2. *Suppose in Theorem 4.1 we assume further that for each $x = (\bar{x}_i)_{i\in I} \in X$, its coordinates $x_i \notin \overline{F_i}(x) \setminus F_i(x)$ for all i in I. Then there exists a point $\bar{x} = (\bar{x}_i)_{i\in I} \in D = \prod_{i\in I} D_i$ such that $\bar{x} \in F_i(\bar{x})$ for each i in I.*

Proof. For each i in I, let $\mathcal{B}_i$ be the collection of all absolutely convex open neighborhoods of zero in E_i and $\mathcal{B} = \prod_{i\in I} \mathcal{B}_i$. Given any $V = \prod_{i\in I} V_i$ in $\mathcal{B}$, let $Q_V = \{x \in D : x_i \in \overline{F_{V_i}}(x) \text{ for all } i \text{ in } I\}$. Then Q_V is a nonempty closed subset of D for each V in $\mathcal{B}$ by Theorem 4.1. Let $\{V^{(1)}, \cdots, V^{(n)}\}$ be any finite subset of $\mathcal{B}$. Write $V^{(i)} = \prod_{j\in I} V_j^{(i)}$, where $V_j^{(i)} \in \mathcal{B}_j$ for each $i = 1, \cdots, n$. Let $V' = \prod_{j\in I}(\bigcap_{i=1}^n V_j^{(i)}) \in \mathcal{B}$. Clearly, $\emptyset \neq Q_{V'} \subseteq \bigcap_{i=1}^n Q_{V^{(i)}}$. Therefore, the family $\{Q_V : V \in \mathcal{B}\}$ has the finite intersection property. Since $Q_V \subset D$ for all V in $\mathcal{B}$ and D is compact, $\bigcap_{V\in\mathcal{B}} Q_V \neq \emptyset$. Let $\bar{x} = (\bar{x}_i)_{i\in I} \in \bigcap_{V\in\mathcal{B}} Q_V$. Then $\bar{x}_i \in \overline{F_{V_i}}(\bar{x})$ for all i in I and all V_i in $\mathcal{B}_i$, i.e., $\bar{x}_i \in \bigcap_{V_i\in\mathcal{B}_i} \overline{F_{V_i}}(\bar{x})$ for all i in I. It follows from Lemma 2.2 that $\bar{x}_i \in \overline{F_i}(\bar{x})$ for all i in I. By assumption, $\bar{x}_i \in F_i(\bar{x})$ for all i in I. $\square$

We remark that if F_i is closed then $x_i \notin \overline{F_i}(x) \setminus F_i(x)$ for each $x = (x_i)_{i\in I}$ in X. As a special case of Theorem 4.2, we have the following collectively Himmelberg type fixed point theorem.

Corollary 4.3. *For each i in a nonempty index set I, let X_i be a nonempty convex subset of a locally convex topological vector space E_i, let D_i be a nonempty compact subset of X_i, and let $f_i : X = \prod_{i\in I} X_i \to D_i$ be a continuous function. Then there exists $\bar{x} = (\bar{x}_i)_{i\in I} \in D = \prod_{i\in I} D_i$ such that $\bar{x} = f_i(\bar{x})$ for each i in I.*

If the index set I is a singleton, then Theorem 4.2 redues to the following corollary, which provides a partial solution to a conjecture of Wu [15].

Corollary 4.4. *Let X be a nonempty convex subset of a locally convex topological vector space E, let D be a nonempty compact subset of X, and let $F : X \to 2^D$ be sub-lower semicontinuous with nonempty convex values. Suppose $x \notin \overline{F}(x) \setminus F(x)$ for each x in X. Then there exists a point $\bar{x}$ in D such that $\bar{x} \in F(\bar{x})$.*

By Theorem 4.1, we have the following almost fixed point theorem.

Corollary 4.5. *The conclusions of Theorems 4.1 and 4.2 remain valid if the condition "$F_i : X \to 2^{D_i}$ is sub-lower semicontinuous for each i in I" is replaced by that "$F_i^{-1}(y_i)$ is open for each y_i in D_i and each i in I."*

Finally we remark that in case I is a singleton, Theorem 4.1 provides a different result from [12, Theorem 3].

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(Lai-Jiu Lin) Department of Mathematics, National Changhua University of Education, Changhua, 50058, Taiwan.

E-mail address: maljlin@math.ncue.edu.tw

(Ngai-Ching Wong) Department of Applied Mathematics, National Sun Yat-sen University, and National Center for Theoretical Sciences, Kaohsiung, 80424, Taiwan.

E-mail address: wong@math.nsysu.edu.tw

(Zenn-Tsuen Yu) Department of Electrical Engineering, Nan-Kai College of Technology and Commerce, Nantour 542, Taiwan.